Minimum Effort Control Using A Variational Approach

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***Abstract*—Modern optimal control theory involves adjoining the equations of motion of a dynamic system to the objective function using dynamic costates; this is done in order to constrain the optimal control solutions to satisfy the equations of motion. The use of costates increases the number of variables and hence increases the complexity of the problem. In this paper a new approach for computing the optimal control for a broad class of problems is presented. This approach adopts a variational approach to derive differential equations for the control. The proposed approach eliminates the need for costates when solving for the control. In this paper, a case study is presented to demonstrate the new approach.**

***Index Terms*—Optimal control, variational methods, costates.**

I. Introduction

The ability to solve Optimal Control Problems (OCPs) in a computationally efficient way is necessary for numerous large scale and complex engineering efforts. There are several methods developed over the past several decades to OCPs that can be broadly categorized into [1] Direct Methods, Indirect Methods, and Dynamic Programming. In a direct method, the state and/or control are approximated and the OCP is transcribed to a nonlinear optimization problem (or nonlinear programming problem) [2]. In a wide range of applications, dynamic programming breaks complex problems into simpler subproblems that are solved recursively as detailed in many references [3]. The Indirect approach provides necessary conditions for optimality, but finding a solution that satisfies these conditions for problems with a significant number of controls, nonlinear models, or distributed parameter systems can be highly challenging. Despite these difficulties, Indirect methods can outperform any other solution method in terms of accuracy [4].

The optimal control theory is a concomitant of the calculus of variations, a branch of mathematical analysis concerned with finding the stationary point of a functional [5]. Roots for OCP can also be found in classical control theory, and in linear and nonlinear programming [6]. There is indeed a wealth of literature that details the modern optimal control theory and how to apply it [7]. The basic process of solving an OCP using this indirect approach involves adjoining the equations of motion of a dynamic system to the objective function using *dynamic costates*; this is done in order to constrain the OCP solutions to satisfy the equations of motion. Through the Euler-Lagrange equations [5], differential equations and terminal conditions are derived for the costates in terms of, in general, the states, control, and time. The result is a two-point boundary value problem in the states and costates that -when solved- returns the optimal trajectories of the system. The Legendre-Clebsch condition determines whether the stationary solution is a minimum or a maximum. The Minimum Principle applies for the more general problem with inequality constraints.

There are few challenges when implementing this current approach for optimal control; one challenge for instance, in many classes of problems, the terminal states are constrained. This shows up for example in space trajectory optimization problems, where it is desired to match the target position and velocity vectors at the final time, whether it be for a planetary rendezvous or orbit insertion. When this occurs, no information is known at the endpoints about the costates, as each of the terminal conditions on the costates is now a Lagrange multiplier to be solved for. Finding a good initial guess for the costates, and subsequently a good solution, is not a trivial task.

A. Contribution

For optimal control problems, where the objective is to minimize the control effort, this paper presents a new approach for optimal control that is different from the modern optimal control theory referred to above. The new approach does not use costates in computing the optimal control, and it computes differential equations describing the optimal time evolution of the control.

II. Problem Statement

Let a physical system to be controlled be described by a set of second order differential equations, and let $x \in \mathbb{R}^{N\times 1}$ be the state vector of the system. The objective of the optimal control problem $\mathcal{J}$ is to satisfy a given set of terminal conditions on the state vector at terminal time, $x(t_f)$, while minimizing the control effort over a given time period $t \in [t_0, t_f]$. It is possible to write the system equations of motion using the generalized coordinates [8] in the form

$$\ddot{q} = f(q, \dot{q}, t) + u \tag{1}$$

where $q \in \mathbb{R}^{N\times 1}$ is the generalized coordinates vector, and $u \in \mathbb{R}^{N\times 1}$ is the generalized control force vector.

The optimal control problem addressed in this letter is the minimum control effort problem. This optimal control problem can be stated as:

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Problem 1:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathcal{J} = \nu^T F(x(t_f), t_f) + \frac{1}{2}\int_{t_0}^{t_f} u^T u \, dt \\ \text{s.t.} \quad & \ddot{q} = f(q, \dot{q}, t) + u \end{aligned} \tag{2}$$

where the terminal constraints vector is $F \in \mathbb{R}^p$ and $\nu \in \mathbb{R}^p$ is the associated vector of Lagrange multipliers.

III. MATH: OPTIMAL CONTROL

First we use the classical optimal control theory to prove a differential equation form for the optimal control problem presented in Section II.

Theorem 1. *For the system defined in* (1)*, the optimal control solution to the minimum control effort problem defined Section II will satisfy*

$$\ddot{u} = -\left(\frac{\partial f}{\partial \dot{q}}\right)^T \dot{u} - \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right) - \frac{\partial f}{\partial q}\right]^T u$$

Proof. Let the state vector $x \in \mathbb{R}^{2n}$ be defined as $x = \begin{bmatrix} q \\ \hline \dot{q} \end{bmatrix}$. Additionally, let the costate vector $\lambda \in \mathbb{R}^{2n}$ be defined as $\lambda = \begin{bmatrix} \Lambda_1 \\ \hline \Lambda_2 \end{bmatrix}$, where $\Lambda_1, \Lambda_2 \in \mathbb{R}^n$ are the costate vectors associated with q and $\dot{q}$, respectively. With these definitions, the optimal control Hamiltonian function for the problem is given by

$$H = \frac{1}{2}\sum u_i^2 + \Lambda_1^T q + \Lambda_2^T (f + u) \tag{3}$$

From the necessary conditions of optimality [7], the costates will have differential equations given by $\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$. For our system,

$$\dot{\Lambda}_1 = -\left(\frac{\partial f}{\partial q}\right)^T \Lambda_2 \tag{4}$$

$$\dot{\Lambda}_2 = -\Lambda_1 - \left(\frac{\partial f}{\partial \dot{q}}\right)^T \Lambda_2 \tag{5}$$

The stationarity condition asserts that $\frac{\partial H}{\partial u} = 0$. For this system, $\frac{\partial h}{\partial u} = u^T + \Lambda_2^T = 0$. This gives the relations

$$\begin{aligned} u &= -\Lambda_2 \\ \dot{u} &= -\dot{\Lambda}_2 \\ \ddot{u} &= -\ddot{\Lambda}_2 \end{aligned} \tag{6}$$

Passing in the relations in Eq. (6) into Eqs. (4) and (5) yields

$$\dot{\Lambda}_1 = \left(\frac{\partial f}{\partial q}\right)^T u \tag{7}$$

$$-\dot{u} = -\Lambda_1 + \left(\frac{\partial f}{\partial \dot{q}}\right)^T u \tag{8}$$

By differentiating Eq. (8) with respect to time and substituting in Eq. (7), we arrive at

$$\ddot{u} = -\left(\frac{\partial f}{\partial q}\right)^T \dot{u} - \left(\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right) - \frac{\partial f}{\partial q}\right)^T u \tag{9}$$

Equation (9) is the set of differential equations that the minimum effort optimal control solution must satisfy. ■

Now that there is a differential equation for the control in terms of only the state and control variables which we know must be satisfied by the optimal control solution, we present a variational approach that finds the same solution found in equation (9) without having to introduce the dynamic costates, and without going through the classical optimal control theory process used in Theorem 1.

In the proposed approach, the optimal control solution can be obtained by minimizing another functional with respect to the vector of generalized coordinates q, as detailed below.

Theorem 2. *The functional $\int_a^b F(t, y, \dot{y})\, dt$ is minimized by the function which satisfies*

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} = 0$$

Proof. The variation of a functional $I = \int G(t, x) dt$ is defined as [9]

$$\delta I = \int \sum \frac{\partial G}{\partial x_i} \delta x_i \, dt$$

The functional will be minimized when $\delta I = 0$. Taking the variation of the functional in this theorem,

$$\delta \int_a^b F(t, y, \dot{y})\, dt = \int_a^b \left(\frac{\partial F}{\partial \dot{y}}\delta \dot{y} + \frac{\partial F}{\partial y}\delta y\right) dt \tag{10}$$

The variations $\delta \dot{y}$ and δy are related, so the equation must be integrated by parts.

$$\int_a^b \frac{\partial F}{\partial \dot{y}} \delta \dot{y}\, dt = \left.\frac{\partial F}{\partial \dot{y}}\delta y\right|_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)\delta y \, dt \tag{11}$$

For many applications, it is common to assume $\delta y(a) = \delta y(b) = 0$ [8]. With this assumption, we can combine Eqs. (10) and (11) to write the total variation of the functional.

$$\delta \int_a^b F(t, y, \dot{y}) dt = -\int_a^b \left(\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y}\right)\delta y \, dt \tag{12}$$

The minimum of this functional will be found when the variation vanishes. The quantity δy is arbitrary and in general nonzero, so the only way to ensure that the integral vanishes is for the bracketed term to be identically zero; that is to say

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial q} = 0 \tag{13}$$

■

Next we use **Theorem 2** to find the solution to **Problem 2** below.

Problem 2: For the dynamic system described by Eq. (1), and considering the function

$$\mathcal{B} = \dot{u}^T \dot{q} + u^T \frac{\partial f}{\partial \dot{q}}\dot{q} - u^T g + u^T h + \frac{1}{2}u^T u \tag{14}$$

where $f = g(q, \dot{q}) + h(q)$, find the minimum of the functional $\int_{t_0}^{t_f} \mathcal{B}\, dt$ over the generalized coordinates q

$$\min_{q} \quad \int_{t_0}^{t_f} \mathcal{B}\, dt \tag{15}$$

Assumption 1. *The system described in Equation* (1) *is autonomous; hence the function f in Eq.* (1) *can be expressed as $f(q, \dot{q})$.*

Theorem 3. *For the class of dynamic systems that can be described using* (1) *and **Assumption 1**, Equation* (9) *is a solution to **Problem 2**.*

Proof. Let $L = T - V$ be the Lagrangian function of the given system, where T is the system's kinetic energy, and V is the system's potential energy. The system equations of motion can be written in the form [10]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \alpha(\ddot{q} - f) \tag{16}$$

where $\alpha = \dfrac{\partial^2 L}{\partial \dot{q}^2}$ is known as the Jacobi Last Multiplier matrix. Note that in the case of an uncontrolled conservative system, $\ddot{q} = f$, and hence the right hand side vanishes. For a conservative system, the position vector will be a function of only the generalized coordinates - that is $r_i = r_i(q)$, for $r_i \in \mathbb{R}^n$. In general, the kinetic energy is of the form $T = \frac{1}{2}\sum m_i |r_i|^2$. From the definition of r,

$$\begin{aligned} \dot{r}_i = \frac{dr_i}{dt} &= \sum_j \frac{\partial r_i}{\partial q_j}\frac{\partial q_j}{\partial t} \\ &= \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j \end{aligned}$$

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \left|\sum_j \frac{\partial r_i}{q_j}\dot{q}_k\right|^2 \\ &= \frac{1}{2}\sum_i m_i \sum_{j,k}\left(\frac{\partial r_i}{\partial q_k}\cdot\frac{\partial r_i}{\partial q_j}\right)\dot{q}_j\dot{q}_k \\ &= \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k \end{aligned} \tag{17}$$

We can write the Lagrangian as

$$L = T - V = \frac{1}{2}\dot{q}^T M \dot{q} - V(q) \tag{18}$$

Note that in general, the mass matrix $M \in \mathbb{R}^{n\times n}$ is positive semidefinite, symmetric, and invertible. From the definition of the Jacobi Last Multiplier matrix α, it is clear that for this class of systems $\alpha = M$. We can use Eq. (16) to find the equations of motion for the system from the Lagrangian function. Computing the partial derivatives of Eq. (18) with respect to q and $\dot{q}$ yields

$$\frac{\partial L}{\partial q} = \frac{1}{2}\frac{\partial}{\partial q}\dot{q}^T M \dot{q} - \frac{\partial V}{\partial q} \tag{19}$$

$$\frac{\partial L}{\partial \dot{q}} = \dot{q}^T M \tag{20}$$

The total time derivative of $\partial L/\partial \dot{q}$ is given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) &= \ddot{q}^T M + \dot{q}^T \dot{M} \\ &= \ddot{q}^T M + \dot{q}^T\left(\sum_i \frac{\partial M}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial M}{\partial t}\right) \\ &= \ddot{q}^T M + \dot{q}^T\left(\sum_i \frac{\partial M}{\partial q_i}\dot{q}_i\right) \end{aligned} \tag{21}$$

From the definition of M in Eq. (17), M is only a function of the generalized coordinates q, hence the above expansion of $\dot{M}$. For the derivative of the Lagrangian with respect to q, we can use the suggestive notation

$$\dot{q}^T\frac{\partial M}{\partial q}\dot{q} = \left[\dot{q}^T\frac{\partial M}{\partial q_1}\dot{q},\ \dot{q}^T\frac{\partial M}{\partial q_2}\dot{q},\ ...,\ \dot{q}^T\frac{\partial M}{\partial q_n}\dot{q}\right]$$

Combining this with Eq. (21) into Eq. (16) yields the final form for the equations of motion of the system.

$$f = M^{-1}\left[\frac{1}{2}\dot{q}^T\frac{\partial M}{\partial q}\dot{q} - \frac{\partial V}{\partial q} - \dot{q}^T\left(\sum_i \frac{\partial M}{\partial q_i}\dot{q}_i\right)\right] \tag{22}$$

Without loss of generality, the function $f(q, \dot{q})$ can be written in the form

$$f = g(q, \dot{q}) + h(q) \tag{23}$$

The g function have the velocity dependent terms in the equations of motion, and h are the strictly position based terms. Therefore,

$$\frac{\partial f}{\partial \dot{q}} = \frac{\partial g}{\partial \dot{q}} + \frac{\partial h}{\partial \dot{q}} = \frac{\partial g}{\partial \dot{q}}$$

From Eq. (22), we can write that

$$g = M^{-1}\left[\frac{1}{2}\dot{q}^T\frac{\partial M}{\partial q}\dot{q} - \dot{q}^T\left(\sum_i \frac{\partial M}{\partial q_i}\dot{q}_i\right)\right]$$

Since this is quadratic in the generalized velocities, it will satisfy the relation

$$\frac{\partial g}{\partial \dot{q}}\dot{q} = 2g \tag{24}$$

To minimize the functional $\int \mathcal{B}\, dt$ with respect to the generalized coordinates q, we will use **Theorem 2**. We first compute the partial derivatives of $\mathcal{B}$ with respect to q and $\dot{q}$.

$$\frac{\partial \mathcal{B}}{\partial q} = u^T\left(\frac{\partial}{\partial q}\frac{\partial f}{\partial \dot{q}}\right)\dot{q} - u^T\frac{\partial g}{\partial q} + u^T\frac{\partial h}{\partial q} \tag{25}$$

$$\frac{\partial \mathcal{B}}{\partial \dot{q}} = \dot{u}^T + u^T\frac{\partial f}{\partial \dot{q}} + u^T\frac{\partial^2 f}{\partial \dot{q}^2}\dot{q} - u^T\frac{\partial g}{\partial \dot{q}} \tag{26}$$

Taking the total derivative of Eq. (26) with respect to time yields

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{B}}{\partial \dot{q}}\right) &= \ddot{u}^T + \dot{u}^T\frac{\partial f}{\partial \dot{q}} + u^T\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right) + \dot{u}^T\frac{\partial^2 f}{\partial \dot{q}^2}\dot{q} \\ &\quad + u^T\frac{d}{dt}\left(\frac{\partial^2 f}{\partial \dot{q}^2}\right)\dot{q} + u^T\frac{\partial^2 f}{\partial \dot{q}^2}\ddot{q} \\ &\quad - \dot{u}^T\frac{\partial g}{\partial \dot{q}} - u^T\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{q}}\right)\end{aligned} \tag{27}$$

By combining Eqs. (27) and (26) and collecting terms, and utilizing **Theorem 2** we arrive at the differential equations for the control from this variational principle.

$$\ddot{u}^T = -\dot{u}^T\frac{\partial^2 f}{\partial \dot{q}^2}\dot{q} - u^T\left[\frac{d}{dt}\left(\frac{\partial^2 f}{\partial \dot{q}^2}\dot{q}\right) - \frac{\partial}{\partial q}\frac{\partial f}{\partial \dot{q}}\dot{q} + \frac{\partial g}{\partial q} - \frac{\partial h}{\partial q}\right] \tag{28}$$

As established above, g (and by extension f) is quadratic in the generalized velocities, and so $\frac{\partial f}{\partial \dot{q}}\dot{q} = \frac{\partial g}{\partial \dot{q}}\partial q = 2g$. Taking the derivative with respect to $\dot{q}$,

$$\begin{aligned}\frac{\partial}{\partial \dot{q}}\left(\frac{\partial f}{\partial \dot{q}}\dot{q}\right) &= \frac{\partial}{\partial \dot{q}}(2g) \\ \frac{\partial^2 f}{\partial \dot{q}^2}\dot{q} + \frac{\partial f}{\partial \dot{q}} &= 2\frac{\partial g}{\partial \dot{q}} \\ \frac{\partial^2 f}{\partial \dot{q}^2}\dot{q} &= \frac{\partial f}{\partial \dot{q}}\end{aligned} \tag{29}$$

The relation $\frac{\partial f}{\partial \dot{q}} = \frac{\partial g}{\partial \dot{q}}$ was used in the above equation. We can use Eq. (29) to rewrite Eq. (28).

$$\begin{aligned}\ddot{u}^T &= -\dot{u}^T\frac{\partial f}{\partial \dot{q}} - u^T\left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right) - \frac{\partial}{\partial q}(2g) + \frac{\partial g}{\partial q} - \frac{\partial h}{\partial q}\right] \\ &= -\dot{u}^T\frac{\partial f}{\partial \dot{q}} - u^T\left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right) - \frac{\partial g}{\partial q} - \frac{\partial h}{\partial q}\right] \\ &= -\dot{u}^T\frac{\partial f}{\partial \dot{q}} - u^T\left[\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right) - \frac{\partial f}{\partial q}\right]\end{aligned} \tag{30}$$

This form for the optimal control is identical to the one in Eq. (9). The minimization of $\int \mathcal{B}\,dt$ therefore yields the correct equations for the optimal control. ■

It is possible to simplify the form of the optimal control functional given in (14) as described below. Define

$$\psi_i(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \tag{31}$$

Following Eq. (16), we can solve for the equations of motion of the uncontrolled system, f, in terms of the Lagrangian, $L = T - V$.

$$\psi(L) = \alpha(\ddot{q} - f) \tag{32}$$

$$\implies f = \ddot{q} - \alpha^{-1}\psi(L) \tag{33}$$

Additionally, the condition in Eq. (24) yields

$$\frac{\partial g}{\partial \dot{q}}\dot{q} = \frac{\partial f}{\partial \dot{q}}\dot{q} = 2g$$

Applying both this and the relation in Eq. (33) to the functional $\mathcal{B}$ in Eq. (14) gives

$$\begin{aligned}\mathcal{B} &= \dot{u}^T\dot{q} + u^T\frac{\partial f}{\partial \dot{q}}\dot{q} - u^T g + u^T h + \frac{1}{2}u^T u \\ &= \dot{u}^T\dot{q} + u^T(2g) - u^T g + u^T h + \frac{1}{2}u^T u \\ &= \dot{u}^T\dot{q} + u^T f + \frac{1}{2}u^T u \\ &= \dot{u}^T\dot{q} + u^T(\ddot{q} - \alpha^{-1}\psi(L)) + \frac{1}{2}u^T u\end{aligned} \tag{34}$$

We can now write the optimal control functional explicitly in terms of the Lagrangian, the control u and the generalized coordinates q.

$$\mathcal{B} = \dot{u}^T\dot{q} + u^T(\ddot{q} - \alpha^{-1}\psi(L)) + \frac{1}{2}u^T u \tag{35}$$

In summary, the new process for finding the optimal control solution for the problem given in **Problem 1**, we start be rearranging the system's equations of motion in the form given in (23) to identify the functions $g(q,\dot{q})$ and $h(q)$. Then we evaluate the respective partial derivatives in (30) for f, g and h with respect to q and $\dot{q}$. Substituting the computed partial derivatives into (30) yields a differential equation for the optimal control solution.

IV. Case Study

To demonstrate the method, we consider the two body problem of gravitation shown in figure 1. The problem describes the position of an orbiting body around a central mass in terms of polar coordinates r, θ. This is a foundational problem in low thrust trajectory design.

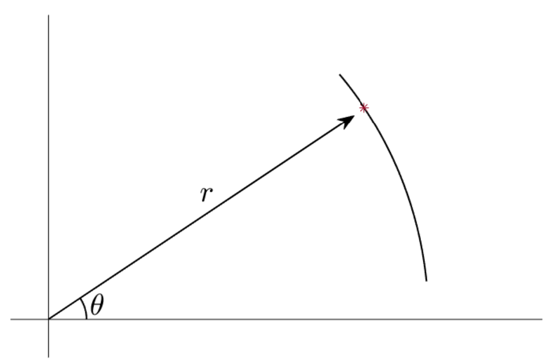

Fig. 1. Polar Two-Body Problem

Suppose the desired control objective is to reach a final prescribed state while minimizing the control. The state vector is given by $x^T = [r, \theta, \dot{r}, \dot{\theta}] = [q^T, \dot{q}^T]$. Formally, the optimal control problem is

$$\min\ \mathcal{J} = \nu^T F + \int_{t_0}^{t_f} \frac{1}{2}(u_r^2 + u_\theta^2)\,dt$$

where $F(t_f, x(t_f)) = 0$ are the terminal state constraints with $F \in \mathbb{R}^p$, $p \leq 4$.

To find the variational principle (Eq. (35)) which will provide the optimal control, we must find the equations of motion and the Jacobi Last Multiplier matrix for the original uncontrolled system. The classical Lagrangian for this system is given by

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu}{r} \tag{36}$$

The Jacobi Last multiplier for this system is

$$\alpha = \frac{\partial^2 L}{\partial \dot{q}^2} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{37}$$

Following Eq. (35), we need to compute $\psi(L)$.

$$\begin{aligned} \psi_1(L) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r} \\ \psi_2(L) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} \end{aligned} \tag{38}$$

With this, we can substitute all values into Eq. (35) to arrive at the functional which, when minimized, will provide the optimal control differential equations.

$$\begin{aligned} \mathcal{B} &= \dot{u}^T\dot{q} + u^T(\ddot{q} - \alpha^{-1}\psi(L)) + \frac{1}{2}u^T u \\ &= \dot{u}_r\dot{r} + \dot{u}_\theta\dot{\theta} + [u_r \ u_\theta]\left(\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} + \right. \\ &\left. - \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}\begin{bmatrix} \ddot{r} - r\dot{\theta}^2 + \mu/r \\ 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} \end{bmatrix}\right) + \frac{1}{2}\left(u_r^2 + u_\theta^2\right) \\ &= \dot{u}_r\dot{r} + \dot{u}_\theta\dot{\theta} + \left(r\dot{\theta}^2 - \frac{\mu}{r^2}\right)u_r - \frac{2\dot{r}\dot{\theta}}{r}u_\theta + \frac{1}{2}\left(u_r^2 + u_\theta^2\right) \end{aligned} \tag{39}$$

Theorem 2 can be applied to minimize this functional with respect to the generalized coordinates $q^T = [r, \theta]$ to arrive at the differential equations of optimal control for this system.

$$\ddot{u}_r = \left(\dot{\theta}^2 + \frac{2\mu}{r^3}\right)u_r + \frac{2\ddot{\theta}}{r}u_\theta + \frac{2\dot{\theta}}{r}\dot{u}_\theta \tag{40}$$

$$\ddot{u}_\theta = -(2\dot{r}\dot{\theta} + 2r\ddot{\theta})u_r + \left(\frac{2\ddot{r}}{r} - \frac{2\dot{r}^2}{r^2}\right)u_\theta - 2r\dot{\theta}^2\dot{u}_r + \frac{2\dot{r}}{r}\dot{u}_\theta \tag{41}$$

It is straightforward to verify that these equations for optimal control are identical to those which would be returned by applying the results of **Theorem 1** to the system.

V. Summary

This paper presents a new approach developed to solve optimal control problems using a variational approach. The new method avoids altogether the use Lagrange multipliers (dynamic costates) when solving for the optimal control solution. This new method does not yield an expression for the control; rather it produces a differential equation for the optimal control. A case study is presented.

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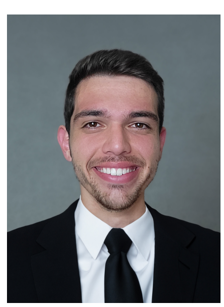

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